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Prime ideals of Bhargava domains

Julie Yeramian

Université Paul Cézanne (Aix-Marseille III), Faculté des Sciences et techniques de St Jérome, avenue Escadrille Normandie-Niemen, 13397 Marseille cedex 20, France

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ABSTRACT

Let D be a domain with quotient field K. For any non-zero $x \in D$, we consider the ring $\mathbb{B}_x(D) = \{f \in K[X] \mid \forall a \in D, f(xX+a) \in D[X]\}$. These domains are subrings of the ring of integer-valued polynomials $\mathrm{Int}(D)$. We study here the prime ideals of $\mathbb{B}_x(D)$ for D a Krull domain. Using some localization properties, we focus first to the case of a discrete valuation domain V, and we determine the prime ideals above $\mathfrak{m}$, the maximal ideal of V. In contrast to $\mathrm{Int}(V)$ where all the primes are of the form $\mathfrak{M}_a = \{f \in \mathrm{Int}(V) \mid f(a) \in \widehat{\mathfrak{m}}\}$, (in one-to-one correspondence with the elements a of $\widehat{V}$ the $\mathfrak{m}$-adic completion of V) we prove that there are two classes of prime ideals above $\mathfrak{m}$ in $\mathbb{B}_x(V)$. We give a complete description of the primes and the maximal ideals of $\mathbb{B}_x(V)$. By globalization, we obtain the prime ideals of $\mathbb{B}_x(D)$ above a height one prime, for D a Krull domain.

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1. Introduction and preliminaries

Let D be a domain with quotient field K. We denote by $\mathrm{Int}(D)$ the ring of integer-valued polynomials over D:

$$\mathrm{Int}(D) = \{f \in K[X] \mid \forall a \in D, f(a) \in D\}.$$

The monograph [4] is a good general reference on the algebraic properties of $\mathrm{Int}(D)$. We consider the following class of subrings of $\mathrm{Int}(D)$: for all $x \in D$, $x \neq 0$, we denote by $\mathbb{B}_x(D)$ the domain defined by

$$\mathbb{B}_x(D) = \{f \in K[X] \mid \forall a \in D, f(xX+a) \in D[X]\}.$$

Introduced by M. Bhargava at the 2nd meeting on the integer-valued polynomials which took place at the C.I.R.M.(Centre International de Rencontres Mathématiques) in June 2000 in Marseille, these domains were the object of the PhD thesis of the current author [10]. A first part of this study has been published in "Anneaux de Bhargava" [11]. We will recall hereafter some of these properties.

The domain $\mathbb{B}_x(D)$ is clearly a subring of $\mathrm{Int}(D)$ containing $D[X]$,

$$D[X] \subset \mathbb{B}_x(D) \subset \mathrm{Int}(D).$$

Noting that $f(xX+a) \in D[X]$ if and only if $f \in D\left[\frac{X-a}{x}\right]$, one gets another description of these domains,

$$\mathbb{B}_x(D) = \bigcap_{a \in D} D\left[\frac{X-a}{x}\right].$$

In fact, since $D\left[\frac{X-a}{x}\right] = D\left[\frac{X-b}{x}\right]$ when $a - b \in xD$, this intersection can be reduced to the intersection on the set of representatives of D modulo xD.

E-mail address: julie.yeramian@univ-cezanne.fr.

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For instance, for $x = 2$ and $\mathbb{Z}$,

$$\mathbb{B}_2(\mathbb{Z}) = \mathbb{Z}\left[\frac{X}{2}\right] \bigcap \mathbb{Z}\left[\frac{X-1}{2}\right].$$

From this description, one can easily check that for any $\lambda \in D$, $\mathbb{B}_x(D) \subseteq \mathbb{B}_{\lambda x}(D)$ and in particular, for u a unit of D, $\mathbb{B}_x(D) = \mathbb{B}_{ux}(D)$.

Moreover, $\mathrm{Int}(D)$ can be recovered from the $\mathbb{B}_x(D)$.

$$\mathrm{Int}(D) = \bigcup_{x \in D, x \neq 0} \mathbb{B}_x(D).$$

1.1. Localization properties

As with the ring of integer-valued polynomials, the domains $\mathbb{B}_x(D)$ have good behaviour under localization. For S a multiplicative subset of D, we have only the inclusion $S^{-1}\mathbb{B}_x(D) \subseteq \mathbb{B}_x(S^{-1}D)$. Under some assumptions, we obtain equality.

Proposition 1.1 (*[11, Propositions 2.2, 2.3 & 2.4]*). *If D is a Noetherian domain and S a multiplicative subset, then $S^{-1}\mathbb{B}_x(D) = \mathbb{B}_x(S^{-1}D)$. If D is a Krull domain and $\mathfrak{p}$ a height one prime of D, then $(\mathbb{B}_x(D))_{\mathfrak{p}} = \mathbb{B}_x(D_{\mathfrak{p}})$. Moreover, we have*

$$\mathbb{B}_x(D) = \bigcap_{\{\mathfrak{p} \mid ht(\mathfrak{p})=1\}} \mathbb{B}_x(D_{\mathfrak{p}}).$$

According to these properties, we can reduce our studies to the local case. Since Dedekind domains or Krull domains are locally discrete valuation domains, we will study the domains $\mathbb{B}_x(V)$, when V is a discrete valuation domain.

1.2. Regular basis and v-orderings of order α

Consider V a discrete valuation domain of quotient field K and denote by v its valuation, $\mathfrak{m}$ the maximal ideal and t a uniformizing parameter (i.e. $\mathfrak{m} = tV$). Let x be a non-zero element of V and α its valuation.

To avoid the case when $\mathrm{Int}(V)$ (and then also $\mathbb{B}_x(V)$) is trivial (i.e. $\mathrm{Int}(V) = \mathbb{B}_x(V) = V[X]$), we assume that the residue field of V is finite [4, Proposition I.3.16]. Set q to be the cardinal of $V/\mathfrak{m}$. Note that in this context, as $\mathbb{B}_x(V) = \mathbb{B}_{ux}(V)$ for u a unit, the domain $\mathbb{B}_x(V)$ only depends on α, the valuation of x.

We recall the construction of a regular basis of $\mathbb{B}_x(V)$ [11, Section 4]. A regular basis of a domain B, $D[X] \subset B \subset K[X]$, is a sequence of polynomials $(f_n)_{n\geq 1}$ of B, such that for all n, $\deg(f_n) = n$ and $(f_n)_{n\geq 1}$ forms a basis of B as a D-module. There are a considerable number of examples of constructions of regular bases for rings of integer-valued polynomials (see [6,7,9]). For $\mathrm{Int}(V)$, such a basis has been built in [3] from the v-orderings defined by M. Bhargava in [2]. Inspired by this construction, we define the notion of v-ordering of order α:

Definition 1.2. Let V be a discrete valuation domain. A *v-ordering of order α* is a sequence $(u_n)_{n\geq 0}$ defined inductively by

- u_0 is an arbitrary chosen element of V
- Given $u_0, \ldots, u_{n-1}$, the nth term u_n is chosen to minimize the expression in a $\sum_{0\leq i\leq n-1} \inf(\alpha, v(a-u_i))$ i.e., for all $a \in V$

$$\sum_{0\leq i\leq n-1} \inf(\alpha, v(u_n - u_i)) \leq \sum_{0\leq i\leq n-1} \inf(\alpha, v(a-u_i)).$$

Given $(u_n)_{n\geq 0}$ a v-ordering of order α, we denote by $w_q^{(\alpha)}(n)$ the obtained minimum

$$w_q^{(\alpha)}(n) = \sum_{0\leq i\leq n-1} \inf(\alpha, v(u_n - u_i)).$$

The sequence $(w_q^{(\alpha)}(n))$ does not depend on the v-ordering of order α chosen. We can compute its values $w_q^{(\alpha)}(n)$ using particular examples of v-orderings [11, Proposition 4.8]

$$w_q^{(\alpha)}(n) = \left[\frac{n}{q}\right] + \left[\frac{n}{q^2}\right] + \cdots + \left[\frac{n}{q^\alpha}\right]$$

and we obtain the following relation

$$w_q^{(\alpha)}(lq^\alpha + r) = l w_q^{(\alpha)}(q^\alpha) + w_q^{(\alpha)}(r). \tag{1}$$

After a careful study, another description of the v-orderings of order α of V can be given [11, Proposition 5.8].

Proposition 1.3. *A sequence $(u_n)_{n\geq 0}$ is a v-ordering of order α if and only if for all j and for all $k \leq \alpha$, the set $\{u_{jq^k}, u_{jq^k+1}, \ldots, u_{jq^k+q^k-1}\}$ is a complete set of representatives of V modulo $\mathfrak{m}^k$.*

From this characterization, it is easy to see that a v-ordering of order α is also a v-ordering of order β, for $\beta \leq \alpha$. Clearly, the V.W.D.W.O. sequences studied by Amice in [1] (see also [4, Section II.2]) are particular v-orderings of order α (for any α).

Consider $(u_n)_{n\geq 0}$ a v-ordering of order α. A regular basis of $\mathbb{B}_x(V)$ is given by the following sequence of polynomials [11, Theorem 4.4],

$$f_0 = 1 \quad \text{and} \quad \text{for } n \geq 1, \quad f_n = \frac{\prod_{i=0}^{n-1}(X-u_i)}{t^{w_q^{(\alpha)}(n)}}.$$

Remark 1.4. Let y be an element of V of valuation β with $\beta < \alpha$. Consider $(u_n)_{n\geq 0}$ a v-ordering of order α. As $(u_n)_{n\geq 0}$ is also a v-ordering of order β, we can form a regular basis (g_n) of the domain $\mathbb{B}_y(V)$ from this sequence.

For $n \leq q^\beta$, we can note that $w_q^{(\alpha)}(n) = w_q^{(\beta)}(n)$. The polynomials $f_0, f_1, \ldots, f_{q^\beta}$ are then equal to the first polynomials of the regular basis (g_n) and in particular, belong also to the domain $\mathbb{B}_y(V)$.

Using relation (1), we obtain the following divisibility relations [11, Lemma 6.1]

- For all $k < \alpha$, and $q^k \leq n < q^{k+1}$, f_n is divisible by f_{q^k} in $\mathbb{B}_x(V)$.
- For $n \geq q^\alpha$, f_n is divisible by f_{q^α}.

Writing the polynomial of $\mathbb{B}_x(V)$ in the regular basis $(f_n)_{n\geq 1}$, we can group the terms divisible by f_{q^k}. Any polynomial f of $\mathbb{B}_x(V)$ can be written in the form:

$$f = f(u_0) + g_0 f_1 + g_1 f_q + \cdots + g_\alpha f_{q^\alpha} \tag{2}$$

with $g_k \in \mathbb{B}_x(V)$ for all k, and

- for $k \leq \alpha - 1$, $d^\circ g_k < q^{k+1} - q^k$
- $g_\alpha = 0$ if $d^\circ f < q^\alpha$, or $d^\circ g_\alpha = d^\circ f - q^\alpha$ if not.

This result implies in particular that $\mathbb{B}_x(V)$ is a finitely generated V-algebra, generated by the $\alpha + 1$ terms of the regular basis $f_{q^k}^{(\alpha)}$, for $0 \leq k \leq \alpha$ [11, Theorem 6.2], that is

$$\mathbb{B}_x(V) = V[f_{q^k}, 0 \leq k \leq \alpha] = V[f_1, f_q, \ldots, f_{q^k}, \ldots, f_{q^\alpha}].$$

By globalization, we obtain that for D a Krull domain, $\mathbb{B}_x(D)$ is a finitely generated D-algebra. As a consequence, if D is an integrally closed Noetherian domain (i.e. a Noetherian Krull domain), the domains $\mathbb{B}_x(D)$ are Noetherian whereas the ring of integer-valued polynomials is not (unless it is trivial).

The aim of this paper is to study the prime ideals of the domains $\mathbb{B}_x(D)$. As we have just seen, this class of rings are particularly interesting when D is a Krull domain. We will then place ourselves in this context throughout this study.

Consider D a Krull domain. For all prime ideal $\mathfrak{p}$ of D, we want to determine the prime ideals of $\mathbb{B}_x(D)$ above $\mathfrak{p}$. First of all, for $\mathfrak{p} = (0)$, since the ideals above (0) lift in $K[X]$, we can easily describe them.

Proposition 1.5. *Let D be a domain with quotient field K and x, a non-zero element of D. The non-zero primes of $\mathbb{B}_x(D)$ above (0) are the ideals $gK[X] \cap \mathbb{B}_x(D)$ where g is an irreducible polynomial of $K[X]$.*

Let $\mathfrak{p}$ be a non-zero prime ideal of D and S a set of representatives of V modulo $\mathfrak{m}$. We recall that $\mathbb{B}_x(D) = \bigcap_{a\in S} D\left[\frac{X-a}{x}\right]$. For any $a \in D$, if $\mathfrak{P}$ is a prime ideal of $D\left[\frac{X-a}{x}\right]$ above $\mathfrak{p}$, $\mathfrak{P}$ contains the ideal $\mathfrak{p}\left[\frac{X-a}{x}\right]$. The intersection $\mathfrak{P} \cap \mathbb{B}_x(D)$ is then a prime ideal of $\mathbb{B}_x(D)$ above $\mathfrak{p}$ containing $\mathfrak{p}\left[\frac{X-a}{x}\right] \cap \mathbb{B}_x(D)$.

We can then ask if they are the only primes of $\mathbb{B}_x(D)$. That is, if each prime ideal of $\mathbb{B}_x(D)$ can be lifted in a domain $D\left[\frac{X-a}{x}\right]$, for some $a \in S$.

Considering the localization properties, we first study this question in the case of a discrete valuation domain V with finite residue field (to avoid the trivial case, $\mathbb{B}_x(V) = V[X]$). We will prove (Remark 2.19) that the prime ideals above $\mathfrak{m}$, the maximal ideal of V, are the (all distinct) traces of the primes above $\mathfrak{m}$ of the domains $V\left[\frac{X-a}{x}\right]$, for all $a \in S$.

It is well known that the prime ideals of $\text{Int}(V)$ above $\mathfrak{m}$ are in one-to-one correspondence with the points of $\widehat{V}$, the $\mathfrak{m}$-adic completion of V [5]. In other words, to any $a \in \widehat{V}$ corresponds the ideal

$$\mathfrak{M}_a = \{f \in \text{Int}(V) \mid f(a) \in \widehat{\mathfrak{m}}\}.$$

These ideals, all maximal, are not finitely generated. The spectrum of $\mathbb{B}_x(V)$ clearly contains the trace of the ideals $\mathfrak{M}_a$, that is

$$\mathfrak{M}_a^{(\alpha)} = \{f \in \mathbb{B}_x(V) \mid f(a) \in \mathfrak{m}\}.$$

These ideals are all maximal, and since $\mathbb{B}_x(V)$ is Noetherian, they are finitely generated. Contrarily to the prime ideals of $\text{Int}(V)$, there are other prime ideals above $\mathfrak{m}$, for instance the ideals

$$\mathfrak{m}_a^{(\alpha)}[X] = \{f \in \mathbb{B}_x(V) \mid f(xX + a) \in \mathfrak{m}[X]\}.$$

We prove that the ideals $\mathfrak{m}_a^{(\alpha)}[X]$ are minimal among the primes containing $\mathfrak{m}$. It allows us to give a complete description of the prime ideals above $\mathfrak{m}$. In Section 3, we determine the maximal ideals of $\mathbb{B}_x(V)$ and in particular, under which conditions a prime above (0) can be maximal. We conclude this study in Section 4, by giving more general results obtained by globalization.

2. Prime ideals of $\mathbb{B}_x(V)$

Let V be a discrete valuation domain with finite residue field. Denote by v, the corresponding valuation on the quotient field K, $\mathfrak{m}$ the maximal ideal of V and t a uniformizing parameter. Let q be the cardinal of the residue field $V/\mathfrak{m}$.

Let x be a non-zero element of V, and α its valuation. Since $\mathbb{B}_x(V)$ is an overring of $V[X]$, its Krull dimension is at most 2 (cf. [8, Section 30]). Moreover, it clearly contains the chain of length 2, $(0) \subset (X) \subset (\mathfrak{m}, X)$. Thus, the domain $\mathbb{B}_x(V)$ is of Krull dimension 2.

The prime ideals of $\mathbb{B}_x(V)$ are above (0) or the maximal ideal $\mathfrak{m}$. As we have seen in Proposition 1.5, the primes above (0) are the ideals $gK[X] \cap \mathbb{B}_x(V)$, for any polynomial g, irreducible in $K[X]$.

Let us determine now the prime ideals above $\mathfrak{m}$. The ideals $\mathfrak{M}_a$ of $\mathrm{Int}(D)$ are not finitely generated and satisfy $\mathrm{Int}(V)/\mathfrak{M}_a \simeq V/\mathfrak{m}$.

2.1. *First class of primes: Study of the ideals $\mathfrak{M}_a^{(\alpha)}$*

For $a \in V$, one can consider $\mathfrak{M}_a^{(\alpha)}$, the trace of the ideal $\mathfrak{M}_a$ on $\mathbb{B}_x(V)$, which is more precisely

$$\mathfrak{M}_a^{(\alpha)} = \mathfrak{M}_a \cap \mathbb{B}_x(V) = \{f \in \mathbb{B}_x(V) \mid f(a) \in m\}.$$

Clearly, we obtain the following result.

Lemma 2.1. *For all $a \in V$, $\mathfrak{M}_a^{(\alpha)}$ is a maximal ideal of $\mathbb{B}_x(V)$ above $\mathfrak{m}$, and $\mathbb{B}_x(V)/\mathfrak{M}_a^{(\alpha)} \simeq V/\mathfrak{m}$.*

As $\mathbb{B}_x(V)$ is Noetherian, these ideals are finitely generated. We determine a set of generators of $\mathfrak{M}_a^{(\alpha)}$ from a particular regular basis of $\mathbb{B}_x(V)$.

Proposition 2.2. *Let $a \in V$. Let $(u_n)_{n\geq 0}$ be a v-ordering of order α with $u_0 = a$ and $(f_n)_{n\geq 1}$ the corresponding regular basis. Then,*

$$\mathfrak{M}_a^{(\alpha)} = (t, f_1, f_q, \dots, f_{q^k}, \dots, f_{q^\alpha}).$$

Proof. Clearly, the ideal $\mathfrak{M}_a^{(\alpha)}$ contains $\mathfrak{m}$. As $u_0 = a$, the polynomials f_{q^k} vanish at a, and therefore, $f_{q^k} \in \mathfrak{M}_a^{(\alpha)}$.

Conversely, according to formula (2), any polynomial $f \in \mathbb{B}_x(V)$ can be written as the sum $f = f(a) + g_0 f_1 + g_1 f_q + \cdots + g_\alpha f_{q^\alpha}$ with $g_k \in \mathbb{B}_x(V)$. If $f \in \mathfrak{M}_a^{(\alpha)}$, then $f(a) \in \mathfrak{m}$ and thus f is in the ideal generated by $t, f_1, f_q, \dots, f_{q^\alpha}$. □

Remark 2.3. In $\mathrm{Int}(V)$, the ideals $\mathfrak{M}_a$ for all $a \in V$, are all distinct. Using the continuity of integer-valued polynomials with respect to the $\mathfrak{m}$-adic topology, one can deduce that they are not finitely generated (cf. [4, V.2.4]).

On the contrary, in $\mathbb{B}_x(V)$, the traces of these ideals are finitely generated. By the same argument, we then see that for a and b close enough to each other, the ideals $\mathfrak{M}_a \cap \mathbb{B}_x(V)$ and $\mathfrak{M}_b \cap \mathbb{B}_x(V)$ are equal. The traces of the ideals $\mathfrak{M}_a$ are not all distinct in $\mathbb{B}_x(V)$ and it is enough to consider $a \in V$ to get all the ideals $\mathfrak{M}_a^{\{\alpha\}}$.

Hence, we want to find out when we have the equality $\mathfrak{M}_a^{(\alpha)} = \mathfrak{M}_b^{(\alpha)}$. To do this, we first establish the following lemma, based on some properties of the v-ordering of order α:

Lemma 2.4. *Let $a \in V$, and $\{u_0, u_1, \dots, u_{q^k-1}\}$ be a set of representatives of V modulo $\mathfrak{m}^k$. If i is the index, $0 \leq i \leq q^k - 1$ such that $v(u_i - a) \geq k$ then,*

$$\frac{\prod_{j=0, j\neq i}^{q^k-1} (u_j - a)}{t^{w_q(q^k)-k}} = (-1)^{w_q(q^k)} \quad \text{modulo } \mathfrak{m}.$$

As a consequence, $\sum_{j=0,j\neq i}^{q^k-1} v(u_j - a) = w_q(q^k) - k$.

Proof. We prove the lemma by induction on k. For $k = 1$, the set $\{u_j - a, 0 \leq j \leq q - 1\}$ is a set of representatives of $V/\mathfrak{m}$. Since the product of the non-zero elements of a field with q elements is equal to -1, we get that $\prod_{0\leq j\leq q-1, j\neq i}(u_j - a) = -1$ modulo $\mathfrak{m}$.

Assume now that the result is true for $k - 1$. The set $\{u_0 - a, \dots, u_{q^k-1} - a\}$ is a set of representatives of $V/\mathfrak{m}^k$, and thus in particular, the union of q sets of representatives of V modulo $\mathfrak{m}^{k-1}$, each one containing an element in the class of a modulo $\mathfrak{m}^{k-1}$.

Applying the inductive assumption to each one of these subsets, we obtain that

$$\frac{\prod_{j=0, j\neq i}^{q^k-1} (u_j - a)}{t^{q(w_q(q^{k-1})-(k-1))}} = (-1)^{q w_q(q^{k-1})} \prod_{j\in I, j\neq i} (u_j - a) \quad \text{mod } \mathfrak{m}$$

where $I = \{0 \le j \le q^k - 1 \mid v(u_j - a) \ge k - 1\}$. We next use the relation $w_q(q^k) = qw_q(q^{k-1}) + 1$, to write that $w_q(q^k) - k = q(w_q(q^{k-1}) - (k-1)) + (q-1)(k-1)$. Then,

$$\frac{\prod_{j=0, j\neq i}^{q^k-1}(u_j - a)}{t^{w_q(q^k)-k}} = (-1)^{qw_q(q^{k-1})} \prod_{j\in I\, j\neq i} \frac{(u_j - a)}{t^{k-1}} \mod \mathfrak{m}.$$

For all $j \in I, j \neq i$, the $(u_j - a)$ are elements of $\mathfrak{m}^{k-1}$, belonging to distinct non-zero classes modulo $\mathfrak{m}^k$. Thus, the set $\{\frac{u_j-a}{t^{k-1}}, j \in I\}$ represents the non-zero classes of V modulo $\mathfrak{m}$. The product of these elements is therefore equal to -1 modulo $\mathfrak{m}$ and the proof is complete. □

Lemma 2.5. *Let $a \in V$, $(u_n)_{n\ge0}$ be a v-ordering of order α with first term $u_0 = a$ and $(f_n)_{n\ge1}$ the corresponding regular basis. For any $b \in V$ and any integer $l \le \alpha$, the following assertions are equivalent:*

(i) $v(b - a) \ge l + 1$
(ii) $f_{q^k}(b) \in \mathfrak{m}$, *for all* $k \le l$.

Proof. For $k \le l$, $f_{q^k}(b) = \frac{(b-u_0)\cdots(b-u_{q^k-1})}{t^{w_q(q^k)}}$ and thus

$$v(f_{q^k}(b)) = \sum_{i=0}^{q^k-1} v(b - u_i) - w_q(q^k).$$

As we have seen in Proposition 1.3, the v-orderings of order α satisfy the assumption of Lemma 2.4. Thus it follows that if $v(b - a) = v(b - u_0) \ge l + 1$, then $v(f_{q^k}(b)) = v(b - a) - k > 0$ for all $k \le l$. Therefore, $f_{q^k}(b) \in \mathfrak{m}$, for all $k \le l$.

Conversely, if $v(b - a) = h \le l$ then we have $v(f_{q^h}(b)) = v(b - a) - h = 0$ (i.e., $f_{q^h}(b) \notin \mathfrak{m}$). □

Proposition 2.6. *Let a, b be two elements of V, and $(u_n)_{n\ge0}$, $(v_n)_{n\ge0}$ be two v-orderings of order α respectively with first term $u_0 = a$ and $v_0 = b$. Let $(f_n)_{n\ge1}$, $(h_n)_{n\ge1}$ be the regular basis corresponding respectively to the sequences $(u_n)_{n\ge0}$ and $(v_n)_{n\ge0}$. For all integers $l \le \alpha$, the following assertions are equivalent :*

(i) $v(b - a) \ge l + 1$.
(ii) $(t, f_1, f_q, \ldots, f_{q^l}) = (t, h_1, h_q, \ldots, h_{q^l})$.

Proof. According to formula (2), for all $k \le l$, we can write

$$f_{q^k} = f_{q^k}(b) + g_0h_1 + g_1h_q + \cdots + g_kh_{q^k}$$

with $g_j \in \mathbb{B}_x(V)$ for all $j \le k$. Therefore, if $f_{q^k}(b) \in \mathfrak{m}$, then $f_{q^k} \in (t, h_1, h_q, \ldots, h_{q^l})$, and conversely, if $f_{q^k} \in (t, h_1, h_q, \ldots, h_{q^l})$, we have $f_{q^k}(b) \in \mathfrak{m}$, since $h_{q^i}(b) = 0$ for all $i \le l$. From Lemma 2.5, the inclusion

$$(t, f_1, f_q, \ldots, f_{q^l}) \subseteq (t, h_1, h_q, \ldots, h_{q^l})$$

is equivalent to $b - a \in \mathfrak{m}^{l+1}$. Interchanging the two basis, we obtain the inverse inclusion under the same condition, and the proof is complete. □

Combining the previous proposition with Proposition 2.2, we obtain

Corollary 2.7. *Let a, b two elements of V. Then, $\mathfrak{M}_a^{(\alpha)} = \mathfrak{M}_b^{(\alpha)}$ if and only if $v(b - a) \ge \alpha + 1$.*

Remarks 2.8. (1) It follows from Proposition 2.6 that the ideal $(t, f_1, f_q, \ldots, f_{q^l})$ generated by the polynomials f_{q^i} obtained from a v-ordering of order α does not depend on this sequence but only on the first term u_0 and more precisely, on its class modulo $\mathfrak{m}^{l+1}$.
(2) Since $\mathbb{B}_{t^{k-1}}(V) \subset \mathbb{B}_{t^k}(V) \subset \mathrm{Int}(V)$, we have

$$(X - a)V[X] + \mathfrak{m} = \mathfrak{M}_a^{(0)} \subset \mathfrak{M}_a^{(1)} \subset \cdots \mathfrak{M}_a^{(k-1)} \subset \mathfrak{M}_a^{(k)} \cdots \subset \mathfrak{M}_a.$$

By Corollary 2.7, we note that the greater k is, the more classes of V modulo $\mathfrak{m}^k$ there are, and thus the more distinct ideals $\mathfrak{M}_a^{(k)}$ too.

For $k = \infty$, that is for $\mathrm{Int}(V)$, they are all distinct: for $a \neq b$, we have $\mathfrak{M}_a \neq \mathfrak{M}_b$.

2.2. Another class of primes: The ideals $\mathfrak{m}_a^{(\alpha)}[X]$

We now describe a new family of prime ideals above $\mathfrak{m}$. For any $a \in V$, we set

$$\mathfrak{m}_a^{(\alpha)}[X] = \{f \in \mathbb{B}_x(V) \mid f(xX + a) \in \mathfrak{m}[X]\}.$$

It is clear that $\mathfrak{m}_a^{(\alpha)}[X]$ is a prime ideal of $\mathbb{B}_x(V)$ above $\mathfrak{m}$ (it is the inverse image of the ideal $\mathfrak{m}[X]$ by the morphism $\phi : \mathbb{B}_x(V) \to V[X]$ defined by $\phi(f) = f(xX + a)$). Moreover, it is easily seen that $\mathfrak{m}_a^{(\alpha)}[X] \subset \mathfrak{M}_a^{(\alpha)}$. Indeed, if $f \in \mathfrak{m}_a^{(\alpha)}[X]$ (i.e. $f(xX + a) \in \mathfrak{m}[X]$) then in particular, $f(a) \in \mathfrak{m}$ (i.e., $f \in \mathfrak{M}_a^{(\alpha)}$). With these notations, we first determine among the generators of $\mathfrak{M}_a^{(\alpha)}$ which belong to $\mathfrak{m}_a^{(\alpha)}[X]$.

Lemma 2.9. *Let $a \in V$, $(u_n)_{n\geq 0}$ be a v-ordering of order α of first term $u_0 = a$ and $(f_n)_{n\geq 1}$ the corresponding regular basis. For all $k \leq \alpha$ we have*

$$v\left(f_{q^k}(xX + a)\right) = \alpha - k.$$

Proof. We examine the valuation of $f_{q^k}(xX + a)$, for $k \leq \alpha$:

$$v\left(f_{q^k}(xX + a)\right) = \sum_{i=0}^{q^k-1} \inf(v(u_i - a), \alpha) - w_q(q^k).$$

Since $u_0 = a$ and $v(u_i - a) < \alpha$ for all $i \neq 0$, we obtain

$$v\left(f_{q^k}(xX + a)\right) = \sum_{i=1}^{q^k-1} v(u_i - a) + \alpha - w_q(q^k).$$

As noted earlier, it is sufficient to apply Lemma 2.4 to obtain the result. $\square$

According to this lemma, among the generators of $\mathfrak{M}_a^{(\alpha)}$, all the polynomials f_{q^k} belong to $\mathfrak{m}_a^{(\alpha)}[X]$, except f_{q^α}. We now deduce the following.

Proposition 2.10. *For all $a \in V$, the inclusion $\mathfrak{m}_a^{(\alpha)}[X] \subset \mathfrak{M}_a^{(\alpha)}$ is strict. In particular, $\mathfrak{M}_a^{(\alpha)}$ is a height 2 prime ideal of $\mathbb{B}_x(V)$.*

We now determine a set of generators of $\mathfrak{m}_a^{(\alpha)}[X]$, similar to the one obtained in Proposition 2.2.

Proposition 2.11. *Let $a \in V$, $(u_n)_{n\geq 0}$ be a v-ordering of order α with first term $u_0 = a$ and $(f_n)_{n\geq 1}$ be the corresponding regular basis. Then,*

$$\mathfrak{m}_a^{(\alpha)}[X] = (t, f_1, f_q, \ldots, f_{q^{\alpha-1}}).$$

Proof. By the previous lemma, the polynomials f_{q^k} belong to $\mathfrak{m}_a^{(\alpha)}[X]$ for $k < \alpha$. Conversely, let us show by induction on the degree n that any polynomial f of $\mathfrak{m}_a^{(\alpha)}[X]$ belongs to the ideal $(t, f_1, f_q, \ldots, f_{q^{\alpha-1}})$. According to (2), we can write

$$f = f(a) + g_0 f_1 + g_1 f_q + \cdots + g_\alpha f_{q^\alpha}$$

where the polynomials g_k are in $\mathbb{B}_x(V)$. Moreover, we know that $g_\alpha = 0$ if $n < q^\alpha$, and otherwise g_α has degree equal to $n - q^\alpha$.

Since $f \in \mathfrak{m}_a^{(\alpha)}[X]$, we have $f(a) \in \mathfrak{m}$, and as an immediate consequence, we get that $f \in (t, f_1, f_q, \ldots, f_{q^{\alpha-1}})$, if $n < q^\alpha$ (as $g_\alpha = 0$). For $n \geq q^\alpha$, as $f, f(a), f_1, f_q, \ldots, f_{q^{\alpha-1}}$ are in $\mathfrak{m}_a^{(\alpha)}[X]$, the polynomial $g_\alpha f_{q^\alpha}$ is in $\mathfrak{m}_a^{(\alpha)}[X]$. But, we have seen in the previous lemma that $f_{q^\alpha} \notin \mathfrak{m}_a^{(\alpha)}[X]$, and so $g_\alpha \in \mathfrak{m}_a^{(\alpha)}[X]$. By the inductive assumption, the polynomial of degree $n - q^\alpha$, g_α is thus in $(t, f_1, f_q, \ldots, f_{q^{\alpha-1}})$ and the result is proved. $\square$

Proposition 2.6 gives an analog of Corollary 2.7 for the ideals $\mathfrak{m}_a^{(\alpha)}[X]$.

Corollary 2.12. *Let $a, b \in V$. The prime ideals $\mathfrak{m}_a^{(\alpha)}[X]$ and $\mathfrak{m}_b^{(\alpha)}[X]$ are equal if and only if $v(a - b) \geq \alpha$.*

2.3. Description of the primes above $\mathfrak{m}$

We now determine all the prime ideals of $\mathbb{B}_x(V)$ above $\mathfrak{m}$. We will see that there exist different primes from the $\mathfrak{M}_a^{(\alpha)}$ or $\mathfrak{m}_a^{(\alpha)}[X]$. We first prove that any prime ideal above $\mathfrak{m}$ contains an ideal $\mathfrak{m}_a^{(\alpha)}[X]$. We further show that it contains only one of these primes.

Proposition 2.13. *Let $\mathfrak{P}$ a prime ideal of $\mathbb{B}_x(V)$ above $\mathfrak{m}$. Then, there exists $a \in V$ such that $\mathfrak{m}_a^{(\alpha)}[X] \subseteq \mathfrak{P}$.*

Proof. For $\alpha = 0$, the result is obvious. Assume then that $\alpha > 0$. We prove by induction on $k < \alpha$ that there exists an element $a \in V$ such that, $(t, f_1, f_q, \ldots, f_{q^k}) \subseteq \mathfrak{P}$, where $(f_n)_{n\geq 1}$ is a regular basis corresponding to a v-ordering of order α with first term a.

It is clear that $t \in \mathfrak{P}$. Consider $(a_0, a_1, \ldots, a_{q-1})$, a set of representatives of V modulo $\mathfrak{m}$. Let $f = \frac{\prod_{i=0}^{q-1}(X-a_i)}{t}$. As $f \in \mathbb{B}_x(V)$, we have $tf = \prod_{i=0}^{q-1}(X - a_i) \in \mathfrak{P}$. Therefore, the prime ideal $\mathfrak{P}$ contains a polynomial $(X - a)$, for an element $a = a_i$.

Taking a v-ordering of order α with first term a, and the corresponding regular basis, we obtain $f_1 = X - a$, and then $(t, f_1) \in \mathfrak{P}$.

Assume that there exists an element $a \in V$ such that $(t, f_1, f_q, \ldots, f_{q^{k-1}}) \subseteq \mathfrak{P}$, where $(f_n)_{n\geq 1}$ is a regular basis corresponding to a v-ordering of order α with first term a. Since the ideal $(t, f_1, f_q, \ldots, f_{q^{k-1}})$ depends only on the first term a, we can choose $(u_n)_{n\geq 0}$ to be a V.W.D.W.O sequence. In this basis we have $f_{q^{k+1}} = \frac{\prod_{i=0}^{q^{k+1}}(X-u_i)}{t^{w_q(q^{k+1})}}$. Since $w_q(q^{k+1}) = qw_q(q^k) + 1$, we can decompose $f_{q^{k+1}}$ as

$$f_{q^{k+1}} = \frac{1}{t}\prod_{j=0}^{q-1} g_j \quad \text{with} \quad g_j = \frac{\prod_{i=jq^k}^{(j+1)q^k-1}(X-u_i)}{t^{w_q(q^k)}}.$$

As the sequences $(u_{jq^k+n})_{n\geq 0}$ are v-orderings of order k, the polynomials g_j are in $\mathbb{B}_{t^k}(V) \subset \mathbb{B}_x(V)$.

On the other hand, $\mathfrak{P}$ contains $tf_{q^{k+1}}$ and thus, there exists an integer j, $0 \leq j \leq q-1$ such that $g_j \in \mathfrak{P}$. For this j, we consider the sequence (u_{jq^k+n}) with first term u_{jq^k} and we denote by $(h_n)_{n\geq 1}$ the corresponding regular basis. As $v(u_0 - u_{jq^k}) \geq k$, Proposition 2.6 implies that

$$(t, f_1, f_q, \ldots, f_{q^{k-1}}) = (t, h_1, h_q, \ldots, h_{q^{k-1}}).$$

Moreover, $h_{q^k} = g_j \in \mathfrak{P}$, and by the inductive assumption, we obtain the inclusion $(t, h_1, h_q, \ldots, h_{q^k}) \subseteq \mathfrak{P}$. □

According to Formula (2) and considering $(f_n)_{n\geq 1}$ a regular basis corresponding to a v-ordering of order α with first term a, any polynomial f of $\mathfrak{P}$ with degree $n < q^\alpha$ can be written as

$$f = f(a) + g_0 f_1 + g_1 f_q + \cdots + g_{\alpha-1} f_{q^{\alpha-1}}$$

where $g_k \in \mathbb{B}_x(V)$, for all k. Since for all $k < \alpha$, the polynomials f_{q^k} belong to $\mathfrak{m}_a^{(\alpha)}[X]$ (Proposition 2.11), we obtain the following.

Corollary 2.14. *Let $\mathfrak{P}$ be a prime ideal of $\mathbb{B}_x(V)$ above $\mathfrak{m}$. Let $a \in V$ such that $\mathfrak{m}_a^{(\alpha)}[X] \subset \mathfrak{P}$. If $f \in \mathbb{B}_x(V)$ is a polynomial of degree $n < q^\alpha$, then $f \in \mathfrak{P}$ if and only if $f \in \mathfrak{m}_a^{(\alpha)}[X]$.*

As previously, we denote by $(f_n)_{n\geq 1}$ the regular basis corresponding to a v-ordering of order α with first term a. Let $\mathfrak{P}$ be a prime ideal of $\mathbb{B}_x(V)$ containing the prime ideal $\mathfrak{m}_a^{(\alpha)}[X] = (t, X-a, \ldots, f_{q^{\alpha-1}})$. From Corollary 2.14, if $\mathfrak{P}$ contains an ideal $\mathfrak{m}_b^{(\alpha)}[X]$, then $\mathfrak{m}_b^{(\alpha)}[X]$ contains the polynomials of degree $n < q^\alpha$, $t, X-a, \ldots, f_{q^{\alpha-1}}$. Then, we would have $\mathfrak{m}_a^{(\alpha)}[X] \subset \mathfrak{m}_b^{(\alpha)}[X]$. Using the fact that the ideals $\mathfrak{m}_a^{(\alpha)}[X]$ are all of height 1 (Proposition 2.10), we obtain the following result.

Corollary 2.15. *Any prime ideal $\mathfrak{P}$ of $\mathbb{B}_x(V)$ above $\mathfrak{m}$ contains one and only one prime $\mathfrak{m}_a^{(\alpha)}[X]$.*

Knowing that each prime above $\mathfrak{m}$ of $\mathbb{B}_x(V)$ contains an ideal $\mathfrak{m}_a^{(\alpha)}[X]$, we can now fix an element $a \in V$ and look for the primes of the quotient $\mathbb{B}_x(V)$ modulo $\mathfrak{m}_a^{(\alpha)}[X]$. As previously, we take $(u_n)_{n\geq 0}$ a v-ordering of order α with first term $u_0 = a$, and $(f_n)_{n\geq 1}$ the corresponding regular basis.

We consider the map Φ which acts on $\mathbb{B}_x(V)$ as $\Phi(f) = f(xX + a)$. It is a homomorphism of the ring $\mathbb{B}_x(V)$ into $V[X]$. Denote by $\overline{\Phi}$ the composition of Φ with the canonical surjection from $V[X]$ into $V/\mathfrak{m}[X]$.

Lemma 2.16. *The homomorphism $\overline{\Phi} : \mathbb{B}_x(V) \to V/\mathfrak{m}[X]$ which associates to $f \in \mathbb{B}_x(V)$, the polynomial $\overline{\Phi}(f) = \overline{f(xX + a)}$ induces an isomorphism $\mathbb{B}_x(V)/\mathfrak{m}_a^{(\alpha)}[X] \simeq V/\mathfrak{m}[X]$.*

Proof. By definition, $\mathfrak{m}_a^{(\alpha)}[X]$ is the kernel of the homomorphism $\overline{\Phi}$. Then, it suffices to prove that $\overline{\Phi}$ is surjective. Let us consider the image of f_{q^α}. We have

$$f_{q^\alpha}(xX + a) = xX\,\frac{\prod_{i=1}^{q^\alpha-1}(xX + a - u_i)}{t^{w_q(q^\alpha)}}.$$

Let u be the coefficient of the term of degree 1 in the polynomial

$$u = x\frac{\prod_{i=1}^{q^\alpha - 1}(a-u_i)}{t^{w_q(q^\alpha)}}.$$

The coefficients of greater degree are sums of products obtained by substituting x for some terms $(a-u_i)$ in u. Since $v(a-u_i) < \alpha = v(x)$, their valuations are strictly greater than $v(u)$. Following Lemma 2.9, $v(u) = v(f_{q^\alpha}(xX+a)) = 0$. Therefore, u is a unit of V and $f_{q^\alpha}(xX+a) = uX$ modulo $\mathfrak{m}[X]$.

The image by $\overline{\Phi}$ then contains X (image of $u^{-1}f_{q^\alpha}$) and the constants (images of constants). It is thus the domain $V/\mathfrak{m}[X]$ and $\overline{\Phi}$ is a surjective homomorphism. □

Remarks 2.17. Note that we have actually proved that

$$f_{q^\alpha}(xX+a) = uX \quad \text{modulo } \mathfrak{m}[X]$$

where $u = \frac{x\prod_{i=1}^{q^\alpha-1}(a-u_i)}{t^{w_q(q^\alpha)}}$. From Lemma 2.4, $u = (-1)^{w_q(q^\alpha)}\frac{x}{t^\alpha}$ modulo $\mathfrak{m}$.

From this fact, we can exhibit the inverse isomorphism: for any $g \in V[X]$, we have

$$\overline{g(X)} = g(\overline{u^{-1}uX}) = g(\overline{u}^{-1}\overline{\Phi}(f_{q^\alpha})) = \overline{\Phi}(g(u^{-1}f_{q^\alpha})).$$

So, the inverse isomorphism is the application which associates to any $\overline{g} \in V/\mathfrak{m}[X]$, the class $g(u^{-1}f_{q^\alpha})$ in $\mathbb{B}_x(V)/\mathfrak{m}_a^{(\alpha)}[X]$, where $u = (-1)^{w_q(q^\alpha)}\frac{x}{t^\alpha}$.

Now, we can give the complete description of the prime ideals above $\mathfrak{m}$.

Theorem 2.18. *The prime ideals of $\mathbb{B}_x(V)$ above $\mathfrak{m}$ are of the following two types*

(1) *For $a \in V$, the ideals*

$$\mathfrak{m}_a^{(\alpha)}[X] = \{f \in \mathbb{B}_x(V) \mid f(xX+a) \in \mathfrak{m}[X]\}.$$

These ideals are non-maximal height one prime ideals, in one-to-one correspondence with the classes of V modulo $\mathfrak{m}^\alpha$. We have $\mathfrak{m}_a^{(\alpha)}[X] = \mathfrak{m}_b^{(\alpha)}[X]$ if and only if $v(a-b) \geq \alpha$.

(2) *Height two maximal ideals, containing one and only one ideal $\mathfrak{m}_a^{(\alpha)}[X]$, and in bijection with the prime ideals of $V[X]$ containing $\mathfrak{m}[X]$.*

To any element a of V and any polynomial $g \in V[X]$ such that $\overline{g}$ is irreducible in $V/\mathfrak{m}[X]$, there corresponds an ideal $\mathfrak{M}_{a,g}^{(\alpha)}$ defined by

$$\mathfrak{M}_{a,g}^{(\alpha)} = \{f \in \mathbb{B}_x(V) \mid f(xX+a) \in \mathfrak{m}[X] + gV[X]\}.$$

Remark 2.19. Using the previous result, one can answer the question asked at the beginning of this article. If S is a set of representatives of $V/\mathfrak{m}^\alpha$, we know that $\mathbb{B}_x(V) = \bigcap_{a\in S} V\left[\frac{X-a}{x}\right]$. We ask if the prime ideals of $\mathbb{B}_x(V)$ above $\mathfrak{m}$ lift in a domain $V\left[\frac{X-a}{x}\right]$, for an element a.

For any $a \in S$, the minimal prime $\mathfrak{m}_a^{(\alpha)}[X]$ is in fact the trace on $\mathbb{B}_x(V)$ of the prime $\mathfrak{m}[\frac{X-a}{x}]$ of $V[\frac{X-a}{x}]$ (i.e., $\mathfrak{m}_a^{(\alpha)}[X] = \mathfrak{m}[\frac{X-a}{x}] \cap \mathbb{B}_x(V)$).

On the other hand, the prime ideals containing $\mathfrak{m}_a^{(\alpha)}[X]$ are of the form

$$\begin{aligned}\mathfrak{M}_{a,g}^{(\alpha)} &= \{f \in \mathbb{B}_x(V) \mid f(xX+a) \in \mathfrak{m}[X] + gV[X]\} \\ &= \left(\mathfrak{m}\left[\frac{X-a}{x}\right] + hV\left[\frac{X-a}{x}\right]\right) \cap \mathbb{B}_x(V)\end{aligned}$$

where $h = g(\frac{X-a}{x})$ is a polynomial of $V[\frac{X-a}{x}]$, irreducible in $V/\mathfrak{m}[\frac{X-a}{x}]$.

Therefore, the primes above $\mathfrak{m}$ are the (all distinct) traces of the prime ideals above $\mathfrak{m}$ of the domains $V[\frac{X-a}{x}]$, for $a \in S$.

To conclude this section, we complete the description of the primes obtained in Propositions 2.2 and 2.11, by giving the set of generators of the maximal ideals of $\mathbb{B}_x(V)$.

Corollary 2.20. *If $a \in V$ and $g \in V[X]$ is an irreducible polynomial of $V/\mathfrak{m}[X]$, then*

$$\mathfrak{M}_{a,g}^{(\alpha)} = \mathfrak{m}_a^{(\alpha)}[X] + g(u^{-1}f_{q^\alpha})\mathbb{B}_x(V) = (t, f_1, \ldots, f_{q^\alpha-1}, g(u^{-1}f_{q^\alpha}))$$

where $u = (-1)^{w_q(q^\alpha)}\frac{x}{t^\alpha}$ is a unit of V and $(f_n)_{n\geq 1}$ is the regular basis corresponding to a v-ordering of order α of first term a.

2.4. Lifting in the domains $\mathbb{B}_x(V)$

The previous results allow us to determine how the prime ideals of $\mathbb{B}_y(V)$ lift in $\mathbb{B}_x(V)$ when y divides x, so that $\mathbb{B}_y(V) \subset \mathbb{B}_x(V)$.

Corollary 2.21. *Let $x, y \in V$ such that $v(y) = \beta$ and $v(x) = \alpha > \beta$. Then, for any prime ideal $\mathfrak{P}$ of $\mathbb{B}_x(V)$ above $\mathfrak{m}$, there exists $a \in V$ such that $\mathfrak{P}$ lifts $\mathfrak{M}_a^{(\beta)}$ in $\mathbb{B}_y(V)$. In particular, all the prime ideals above $\mathfrak{m}$ of $\mathbb{B}_x(V)$ lift the maximal ideal $\mathfrak{m} + (X - a)V[X]$ of $V[X]$, for some $a \in V$.*

Proof. According to Proposition 2.13, $\mathfrak{P}$ contains an ideal of the form $\mathfrak{m}_a^{(\alpha)}[X]$ in $\mathbb{B}_x(V)$. Since $\mathfrak{M}_a^{(\beta)}$ is maximal in $\mathbb{B}_y(V)$, it suffices to show $\mathfrak{m}_a^{(\alpha)}[X] \cap \mathbb{B}_y(V) = \mathfrak{M}_a^{(\beta)}$. Consider a v-ordering of order α, (thus, also of order β), and $(f_n)_{n\geq 1}$ the corresponding regular basis of $\mathbb{B}_x(V)$. It follows from Proposition 2.11 that $\mathfrak{m}_a^{(\alpha)}[X] = (t, f_1, f_q, \ldots, f_{q^{\alpha-1}})$. As $\alpha > \beta$, the ideal $\mathfrak{m}_a^{(\alpha)}[X] \cap \mathbb{B}_y(V)$ contains t and the f_{q^k} for $k \leq \beta$, which belong to $\mathbb{B}_y(V)$ (see Remark 1.4). Using Proposition 2.2, we then conclude that $\mathfrak{M}_a^{(\beta)} = (t, f_1, f_q, \ldots, f_{q^\beta}) = \mathfrak{m}_a^{(\alpha)}[X] \cap \mathbb{B}_y(V)$. $\square$

Remark 2.22. Knowing how the primes lift in the domains $\mathbb{B}_x(V)$, one can explain why the completion $\widehat{V}$ of V occurs in the description of the prime ideals above $\mathfrak{m}$ of $\mathrm{Int}(V)$. Consider $\mathfrak{P}$, a prime ideal of $\mathrm{Int}(V)$. The domain $\mathrm{Int}(V)$ is the union of the increasing sequence of the domains $\mathbb{B}_{t^k}(V)$,

$$\mathrm{Int}(V) = \bigcup_{k\geq 0} \mathbb{B}_{t^k}(V).$$

Then, for all $k \geq 0$, $\mathfrak{P} \cap \mathbb{B}_{t^k}(V)$ is a prime ideal of $\mathbb{B}_{t^k}(V)$ which lifts in $\mathbb{B}_{t^{k+1}}$. According to Corollary 2.21, the traces $\mathfrak{P} \cap \mathbb{B}_{t^k}(V)$ are of the form $\mathfrak{P} \cap \mathbb{B}_{t^k}(V) = \mathfrak{M}_{a_k}^{(k)}$ for an element $a_k \in V$. On another side, as $\mathbb{B}_{t^l}(V) \subset \mathbb{B}_{t^k}(V)$ for all integers $k > l$, we have

$$\mathfrak{M}_{a_l}^{(l)} = \mathfrak{P} \cap \mathbb{B}_{t^l}(V) = \mathfrak{M}_{a_k}^{(k)} \cap \mathbb{B}_{t^l}(V) = \{f \in \mathbb{B}_{t^l}(V) \mid f(a_k) \in \mathfrak{m}\} = \mathfrak{M}_{a_k}^{(l)}.$$

From Corollary 2.7, the sequence (a_k) is then a Cauchy sequence since $a_k - a_l \in \mathfrak{m}^{l+1}$ for all $k > l$. We can check that $\mathfrak{P} = \mathfrak{M}_a$, where $a \in \widehat{V}$ is the limit of this sequence.

Indeed, as $\mathrm{Int}(V) = \cup_{k\geq 0} \mathbb{B}_{t^k}(V)$, we have

$$\mathfrak{P} = \bigcup_{k\geq 0} (\mathfrak{P} \cap \mathbb{B}_{t^k}(V)) = \bigcup_{k\geq 0} \mathfrak{M}_{a_k}^{(k)}.$$

Let $f \in \mathfrak{P} = \cup_{k\geq 0} \mathfrak{M}_{a_k}^{(k)}$. There exists an integer k_0 such that $f \in \mathfrak{M}_{a_k}^{(k)}$, for all $k \geq k_0$. So, for all $k \geq k_0$, we get $f(a_k) \in \mathfrak{m}$, that is $v(f(a_k)) \geq 1$. By continuity of f and of the valuation v, the limit $v(f(a))$ is greater than 1 and $f(a) \in \widehat{\mathfrak{m}}$. Therefore, $f \in \mathfrak{M}_a$. Conversely, if $f \in \mathfrak{M}_a$ then $v(f(a)) \geq 1$. The sequence $v(f(a_k))$ is a sequence of integers of limit $v(f(a)) \geq 1$. Thus, there exists a rank k_0 such that $v(f(a_k)) \geq 1$, for $k \geq k_0$. Moreover, as $f \in \mathrm{Int}(V) = \cup_{k\geq 0} \mathbb{B}_{t^k}(V)$, there exists an integer k_1 from which $f \in \mathbb{B}_{t^k}(V)$. Therefore, considering $k = \max(k_0, k_1)$, we obtain that $f \in \mathfrak{M}_{a_k}^{(k)} \subset \mathfrak{P}$.

3. Maximal ideals

We have seen previously (Theorem 2.18) that among the prime ideals above $\mathfrak{m}$, the only maximal ones are the primes $\mathfrak{M}_{a,g}^{(\alpha)}$

$$\mathfrak{M}_{a,g}^{(\alpha)} = \{f \in \mathbb{B}_x(V) \mid f(xX + a) \in \mathfrak{m}[X] + gV[X]\}$$

where $a \in V$ and $g \in V[X]$ such that $\overline{g}$ is irreducible in $V/\mathfrak{m}[X]$.

In order to know the set of all maximal ideals of $\mathbb{B}_x(V)$, it remains to examine under which conditions a prime ideal above (0) is maximal.

Let g be an irreducible polynomial of $K[X]$. We denote by $\widehat{g}$ the ideal $\widehat{g} = gK[X] \cap \mathbb{B}_x(V)$. Recall that the ideal generated by g in $\mathrm{Int}(V)$, $gK[X] \cap \mathrm{Int}(V)$ is maximal if and only if g does not have a root in the completion $\widehat{V}$ of V. Indeed, if g has a root $\hat{a}$ in $\widehat{V}$, then clearly the ideal of $\mathrm{Int}(V)$, $gK[X] \cap \mathrm{Int}(V)$ is contained in $\mathfrak{M}_{\hat{a}}$. Taking the intersection with $\mathbb{B}_x(V)$ and using Corollary 2.7, we obtain

$$\widehat{g} \subset \mathfrak{M}_{\hat{a}} \cap \mathbb{B}_x(V) = \mathfrak{M}_a^{\alpha},$$

for $a \in V$ with the same class of $\hat{a}$ modulo $\widehat{\mathfrak{m}}^{\alpha+1}$. This condition is then also necessary for $\widehat{g}$ to be maximal. But, we will see that it is not sufficient.

If g is a polynomial of $K[X]$, for all $a \in V$, $g(a)$ is the constant term of $g(xX + a)$, and then

$$v\left(g(xX + a)\right) \leq v\left(g(a)\right).$$

The following Lemma asserts that a necessary condition for $\widehat{g}$ to be maximal, is the equality $v\left(g(xX + a)\right) = v\left(g(a)\right)$.

Lemma 3.1. *Let g be an irreducible polynomial of $K[X]$ and $a \in V$. If*

$$v\left(g(xX+a)\right) < v\left(g(a)\right)$$

then $\widehat{g}$ is contained in the ideal $\mathfrak{M}_a^{(\alpha)}$ of $\mathbb{B}_x(V)$.

Proof. Each element of $\widehat{g}$ is of the form gh where $h \in K[X]$ and $gh \in \mathbb{B}_x(V)$. By hypothesis, we have

$$v\left(gh(a)\right) = v\left(g(a)\right) + v\left(h(a)\right) > v\left(g(xX+a)\right) + v\left(h(xX+a)\right) = v\left(gh(xX+a)\right).$$

But, by definition of $\mathbb{B}_x(V)$, $gh(xX+a)$ is a polynomial of $V[X]$ (i.e., $v\left(gh(xX+a)\right) \geq 0$). We then deduce that $gh(a) \in \mathfrak{m}$, that is $gh \in \mathfrak{M}_a$. $\square$

In order to $\widehat{g}$ to be maximal, the equality $v\left(g(xX+a)\right) = v\left(g(a)\right)$ has to be satisfied, for all $a \in V$. Under this condition, g has no root in $\widehat{V}$. Indeed, for g of degree n and of leading coefficient a_n, $v\left(g(xX+a)\right)$ is bounded by $v(a_n x^n)$ and a fortiori $v\left(g(a)\right)$ is also bounded. On the contrary, the following example shows that the condition that g has no root in $\widehat{V}$ is not sufficient to ensure $\widehat{g}$ to be maximal.

Example 3.2. Let $g = X^2 + t^3$. Then, g has no root in $\widehat{V}$. But, for $\alpha = 1$ and $a = 0$, we have $v\left(g(xX+a)\right) = 2$, and $v\left(g(a)\right) = 3$. It follows by the lemma that $\widehat{g}$ is contained in $\mathfrak{M}_0$.

Since $g(xX+a) = (g(xX+a) - g(a)) + g(a)$, we have $v\left(g(xX+a)\right) < v\left(g(a)\right)$ if and only if $v\left(g(xX+a) - g(a)\right) < v\left(g(a)\right)$. We have seen that this inequality implies that $\widehat{g}$ is contained in $\mathfrak{M}_a$. Now, we prove in the next lemma, that this large inequality $v\left(g(xX+a) - g(a)\right) \leq v\left(g(a)\right)$ implies that $\widehat{g}$ is contained in a maximal ideal.

Remark 3.3. As the proof is based on the fact that the domain $\mathbb{B}_x(V)$ is a subring of $V\left[\frac{X-a}{x}\right]$, we will write the polynomial g in the form $g = a_0 + a_1\left(\frac{X-a}{x}\right) + \cdots + a_n\left(\frac{X-a}{x}\right)^n$, with the coefficients $a_i \in K$. Note that $g(a) = a_0$ and $g(xX+a) = a_0 + a_1X + \cdots + a_nX^n$. So, determining that g satisfies the inequality $v\left(g(xX+a) - g(a)\right) \leq v\left(g(a)\right)$, is equivalent to saying that there exists an index $i \geq 1$, for which $v(a_i) = \inf_{j\geq 0} v(a_j) = v\left(g(xX+a)\right)$.

Lemma 3.4. *Let g be an irreducible polynomial of $K[X]$ and $a \in V$. If*

$$v\left(g(xX+a) - g(a)\right) \leq v\left(g(a)\right)$$

then $\widehat{g}$ is contained in a maximal ideal containing $\mathfrak{m}_a^{(\alpha)}[X]$.

Proof. As in Remark 3.3, we write g in the form $g = a_0 + a_1\left(\frac{X-a}{x}\right) + \cdots + a_n\left(\frac{X-a}{x}\right)^n$, with $a_j \in K$ for all j. By hypothesis, we know that there exists $i \geq 1$ such that $v(a_i) = \inf_{j\geq 0} v(a_j) = v\left(g(xX+a)\right)$. Set $h = \frac{1}{a_i}g$. It is easy to check that h is a polynomial of $V\left[\frac{X-a}{x}\right]$, of valuation 0 and non-constant modulo $\mathfrak{m}$ (indeed, its ith coefficient is equal to 1). Moreover, we have $\widehat{g} = \widehat{h}$.

The ideal $\widehat{h}$ is lifted in $V\left[\frac{X-a}{x}\right]$ by the prime ideal $hK[X] \cap V\left[\frac{X-a}{x}\right]$. Since, h is of valuation 0, we get $hK[X] \cap V\left[\frac{X-a}{x}\right] = hV\left[\frac{X-a}{x}\right]$. On the other hand, h is not a constant polynomial modulo $\mathfrak{m}$. We can then choose h', a polynomial of $V\left[\frac{X-a}{x}\right]$, which is an irreducible factor of h modulo $\mathfrak{m}\left[\frac{X-a}{x}\right]$. The prime ideal $hV\left[\frac{X-a}{x}\right]$ is thus contained in the prime ideal above $\mathfrak{m}$, containing $\mathfrak{m}\left[\frac{X-a}{x}\right]$ and generated by the polynomial h'. Hence,

$$hV\left[\frac{X-a}{x}\right] \subset \mathfrak{m}\left[\frac{X-a}{x}\right] + h'V\left[\frac{X-a}{x}\right].$$

Taking the intersection with $\mathbb{B}_x(V)$, according to Remark 2.19, we obtain that $\widehat{g}$ is contained in $\mathfrak{M}_{a,h'(xX+a)}^{(\alpha)}$, a maximal ideal above $\mathfrak{m}$ which contains $\mathfrak{m}_a^{(\alpha)}[X]$. $\square$

Remark 3.5. Let g be an irreducible polynomial of $K[X]$ such that $v\left(g(xX+a) - g(a)\right) \leq v\left(g(a)\right)$. Note that we have actually proved that $\widehat{g}$ is contained in all the maximal ideals $\mathfrak{M}_{a,h'(xX+a)}^{(\alpha)}$, for any polynomial h' of $V\left[\frac{X-a}{x}\right]$, which is an irreducible factor of $h = \frac{1}{a_i}g$ modulo $\mathfrak{m}\left[\frac{X-a}{x}\right]$.

Now, we prove that the conditions of the previous lemmas are necessary and sufficient.

Proposition 3.6. *Let g be an irreducible polynomial of $K[X]$ and $a \in V$. Then,*

(i) *$\widehat{g}$ is contained in a maximal ideal containing $\mathfrak{m}_a[X]$ if and only if $v\left(g(xX+a) - g(a)\right) \leq v\left(g(a)\right)$.*
(ii) *$\widehat{g}$ is contained in the ideal $\mathfrak{M}_a^{(\alpha)}$ if and only if $v\left(g(xX+a)\right) < v\left(g(a)\right)$.*

Proof. As previously noted (see Remark 3.3), we write $g = a_0 + a_1\left(\frac{X-a}{x}\right) + \cdots + a_n\left(\frac{X-a}{x}\right)^n$. It remains to show, on the one hand that $v\left(g(xX+a) - g(a)\right) > v\left(g(a)\right)$ (that is $v(a_i) > v(a_0)$ for any $1 \leq i \leq n$) implies that $\widehat{g}$ is not contained in a maximal ideal containing $\mathfrak{m}_a^{(\alpha)}[X]$ and on the other hand that $v\left(g(xX+a)\right) = v\left(g(a)\right)$ (i.e. $v(a_ix^i) \geq v(a_0)$ for $1 \leq i \leq n$) implies that $\widehat{g}$ is not contained in $\mathfrak{M}_a^{(\alpha)}$.

Let $(u_n)_{n\geq 0}$ be a V.W.D.W.O sequence with first term $u_0 = a$ and $(f_n)_{n\geq 1}$ be the associated regular basis. Using the expression $w_q(q^\alpha) = w_q(q^\alpha - 1) + \alpha$, one can write

$$f_{q^\alpha} = \frac{X-a}{t^\alpha} h, \quad \text{where} \quad h = \frac{\prod_{i=1}^{q^\alpha - 1}(X - u_i)}{t^{w_q(q^\alpha - 1)}}.$$

Since the shifted sequence (u_n), $n \geq 1$, is still a V.W.D.W.O sequence, the polynomial h belongs to $\mathbb{B}_x(V)$. On the other hand, it does not belong to $\mathfrak{m}_a^{(\alpha)}[X]$, that is, $h(xX + a)$ does not belong to $\mathfrak{m}[X]$. Indeed,

$$v\left(h(xX+a)\right) = \sum_{i=1}^{q^\alpha - 1} \inf\left(v\left(u_i - u_0\right), \alpha\right) - w_q(q^\alpha - 1) = \sum_{i=1}^{q^\alpha - 1} v_q(i) - w_q(q^\alpha - 1) = 0.$$

As h is of degree $q^\alpha - 1 < q^\alpha$ we can even deduce that it does not belong to any maximal ideal containing $\mathfrak{m}_a^{(\alpha)}[X]$ (Corollary 2.14).
To simplify the notation, we can assume that $x = t^\alpha$, and thus $(X - a)h = x f_{q^\alpha}$. Consider then the product

$$\frac{h^n}{a_0}.g = \frac{h^n}{a_0}.\left(a_0 + a_1\left(\frac{X-a}{x}\right) + \cdots + a_n\left(\frac{X-a}{x}\right)^n\right).$$

Replacing $\left(\frac{X-a}{x}\right)^i h^i$ by $\left(f_{q^\alpha}\right)^i$, we get,

$$\frac{h^n}{a_0}.g = h^n + \frac{a_1}{a_0} f_{q^\alpha} h^{n-1} + \cdots + \frac{a_i}{a_0}\left(f_{q^\alpha}\right)^i h^{n-i} + \cdots + \frac{a_n}{a_0}\left(f_{q^\alpha}\right)^n.$$

Under either of the hypotheses, we have $v(a_i) \geq v(a_0)$ (i.e., $\frac{a_i}{a_0} \in V$ for $1 \leq i \leq n$). Each term on the right-hand side of the equation is thus in $\mathbb{B}_x(V)$ and the product $\frac{h^n}{a_0} g$ is contained in $\widehat{g}$. We can then conclude in both cases:

(i) Let $\mathfrak{M}$ be a maximal ideal containing $\mathfrak{m}_a^{(\alpha)}[X]$. We assume that $v(a_i) > v(a_0)$, and then $\frac{a_i}{a_0} \in \mathfrak{m}$, for any $1 \leq i \leq n$. The terms of the form $\frac{a_i}{a_0}\left(f_{q^\alpha}\right)^i h^{n-i}$ are in $\mathfrak{M}$. But, we have seen that h, and then also h^n, are not in $\mathfrak{M}$. So, $\frac{h^n}{a_0} g \notin \mathfrak{M}$, and $\widehat{g}$ is not contained in $\mathfrak{M}$.

(ii) Consider now the ideal $\mathfrak{M}_a^{(\alpha)}$. As $f_{q^\alpha} \in \mathfrak{M}_a^{(\alpha)}$, the terms of the form $\frac{a_i}{a_0}\left(f_{q^\alpha}\right)^i h^{n-i}$ are in $\mathfrak{M}_a^{(\alpha)}$. But, as in the previous case h is not. So, $\frac{h^n}{a_0} g \notin \mathfrak{M}_a^{(\alpha)}$ and $\widehat{g}$ is not contained in $\mathfrak{M}_a^{(\alpha)}$. □

Corollary 3.7. *Let g be an irreducible polynomial of $K[X]$. Then, $\widehat{g}$ is maximal if and only if $v\left(g(xX + a) - g(a)\right) > v\left(g(a)\right)$, for all $a \in V$.*

We can assume, by multiplying by a constant if necessary, that g is a polynomial of $V[X]$. We then find a more natural sufficient condition, (but which is not necessary) for $\widehat{g}$ to be maximal.

Corollary 3.8. *Let g be a polynomial of $V[X]$, irreducible in $K[X]$. If, for all $a \in V$, we have $v\left(g(a)\right) \leq \alpha$ then $\widehat{g}$ is maximal in $\mathbb{B}_x(V)$.*

Proof. By contradiction, let us assume that $\widehat{g}$ is contained in a maximal ideal $\mathfrak{M}$. There exists $a \in V$ such that $\mathfrak{M}$ contains $\mathfrak{m}_a[X]$. We can write g in the form $g = a_0 + a_1(X - a) + \cdots + a_n(X - a)^n$. The polynomial g is in $\widehat{g}$, thus in $\mathfrak{M} \cap V[X] = \mathfrak{M}_a$. Therefore, $g(a) = a_0 \in \mathfrak{m}$. Considering everything modulo $\mathfrak{m}$, we get

$$\overline{g} = \overline{a_1}(X - \overline{a}) + \cdots + \overline{a_n}(X - \overline{a})^n.$$

We claim that $a_1 \in \mathfrak{m}$. Otherwise, a would be a simple root of g modulo $\mathfrak{m}$ and g would have a root in $\widehat{V}$ (according to Hensel's Lemma). But, then $v\left(g(a)\right)$ would not be bounded over V which contradicts the hypothesis.

As $a_1 \in \mathfrak{m}$, we obtain a contradiction with the previous proposition, noting that $v\left(g(xX + a) - g(a)\right) > v\left(g(a)\right)$. Indeed, we have $v(a_0) \leq \alpha$, $v(a_1 x) \geq 1 + \alpha$ and $v(a_i x^i) \geq 2\alpha$, for all $i \geq 2$. □

Example 3.9. Let $g = X^{2r} + t^{2r-1}$. We can check that g satisfies the condition $v\left(g(xX + a) - g(a)\right) > v\left(g(a)\right)$, for all $a \in V$. On the opposite side, $v\left(g(0)\right) = 2r - 1$ is greater than α for r big enough.

4. Krull domains and Noetherian domains

In this section, we will try to generalize the previous results to larger classes of domains. First, by globalization, the complete description of primes obtained for valuation domains allows us to determine all the primes of $\mathbb{B}_x(D)$ above a height one prime of a Krull domain D and thus, in particular all prime ideals of $\mathbb{B}_x(D)$ for D a Dedekind domain.

Consider D a Krull domain and $\mathfrak{P}$ a prime ideal of $\mathbb{B}_x(D)$ above a height one prime $\mathfrak{p}$ of D. If $\mathfrak{p}$ is a prime ideal which does not contain x, or with infinite residue field, the prime ideals above $\mathfrak{p}$ lift in the domain $(\mathbb{B}_x(D))_{\mathfrak{p}} = D_{\mathfrak{p}}[X]$. Therefore, we get the following result.

Proposition 4.1. *Let $\mathfrak{p}$ be a height one prime ideal of D which does not contain x or has infinite residue field. Then, the prime ideals of $\mathbb{B}_x(D)$ above $\mathfrak{p}$ are:*

(1) *The prime ideal $\mathfrak{p}D_{\mathfrak{p}}[X] \cap \mathbb{B}_x(D)$, that is the set of polynomials of $\mathbb{B}_x(D)$ with coefficients in $\mathfrak{p}D_{\mathfrak{p}}[X]$.*
(2) *For any polynomial g of $D_{\mathfrak{p}}[X]$ irreducible modulo $\mathfrak{p}D_{\mathfrak{p}}[X]$, the prime ideals $(\mathfrak{p}, g)D_{\mathfrak{p}}[X] \cap \mathbb{B}_x(D)$, that is the set of polynomials of $\mathbb{B}_x(D)$ divisible by g modulo $\mathfrak{p}D_{\mathfrak{p}}[X]$.*

So, let us assume now that $\mathfrak{p}$ is a prime ideal containing x and with finite residue field. According to the localization properties, we have $[\mathbb{B}_x(D)]_{\mathfrak{p}} = \mathbb{B}_x(D_{\mathfrak{p}})$ with $D_{\mathfrak{p}}$ is a discrete valuation domain. In this case, the primes of $\mathbb{B}_x(D)$ above $\mathfrak{p}$ are given by Theorem 2.18. We can then describe all the prime ideals of $\mathbb{B}_x(D)$ above a height one prime $\mathfrak{p}$ of D. In particular, for a one-dimensional Krull domain, that is a Dedekind domain, this theorem gives all the primes of $\mathbb{B}_x(D)$.

Theorem 4.2. *Let D be a Krull domain and $x \in D$. Let $\mathfrak{p}$ be a height one prime of D. Denote by α, the valuation of x in $D_{\mathfrak{p}}$. The prime ideals of $\mathbb{B}_x(D)$ above $\mathfrak{p}$ are of the following types.*

(1) *If $\mathfrak{p}$ does not contain x or if $\mathfrak{p}$ has an infinite residue field, the prime ideals above $\mathfrak{p}$ are the ideals:*
  – $\mathfrak{p}D_{\mathfrak{p}}[X] \cap \mathbb{B}_x(D)$,
  – $(\mathfrak{p}, g)D_{\mathfrak{p}}[X] \cap \mathbb{B}_x(D)$, *for any polynomial g of $D_{\mathfrak{p}}[X]$ irreducible modulo $\mathfrak{p}D_{\mathfrak{p}}[X]$.*
(2) *If $\mathfrak{p}$ is a prime ideal containing x and with finite residue field, the prime ideals above $\mathfrak{p}$ are the ideals:*
  – *the ideals $\mathfrak{p}_a^{(\alpha)}[X]$, in one-to-one correspondence with the classes of $D_{\mathfrak{p}}$ modulo $\mathfrak{p}^{\alpha}D_{\mathfrak{p}}$: to each $a \in D_{\mathfrak{p}}$ corresponds*
$$\mathfrak{p}_a^{(\alpha)}[X] = \{f \in \mathbb{B}_x(D) \mid f(xX + a) \in \mathfrak{p}D_{\mathfrak{p}}[X]\} = \mathfrak{p}D_{\mathfrak{p}}\left[\frac{X-a}{x}\right] \cap \mathbb{B}_x(D).$$
  – *the ideals $\mathfrak{P}_{a,g}$ containing one and only prime $\mathfrak{p}_a^{(\alpha)}[X]$, in one-to-one correspondence with the polynomials g of $D_{\mathfrak{p}}[X]$, irreducible modulo $\mathfrak{p}D_{\mathfrak{p}}[X]$*
$$\mathfrak{P}_{a,g} = \{f \in \mathbb{B}_x(D) \mid f(xX + a) \in (\mathfrak{p}, g)D_{\mathfrak{p}}[X]\} = \left(\mathfrak{p}, g\left(\frac{X-a}{x}\right)\right) D_{\mathfrak{p}}\left[\frac{X-a}{x}\right] \cap \mathbb{B}_x(D).$$

In the case of a discrete valuation domain, the description of the prime ideals above the maximal ideal $\mathfrak{m}$ is based on two main results:

– The prime ideals of $\mathbb{B}_x(V)$ above $\mathfrak{m}$ can be lifted in a domain $V[\frac{X-a}{x}]$, and thus contain a prime of the form $\mathfrak{m}[\frac{X-a}{x}] \cap \mathbb{B}_x(D)$ (Proposition 2.13).
– The homomorphism $\overline{\Phi} : \mathbb{B}_x(V) \to V/\mathfrak{m}[X]$ which associates to $f \in \mathbb{B}_x(V)$, $\overline{\Phi}(f) = \overline{f(xX + a)}$ induces an isomorphism from $\mathbb{B}_x(V)/(\mathfrak{m}[\frac{X-a}{x}] \cap \mathbb{B}_x(V))$ to $V/\mathfrak{m}[X]$ (Lemma 2.16).

We can establish a result analogous to Proposition 2.13 in the case of a Noetherian domain. Let S be a set of representatives of D/xD. Consider the intersection ideal, $I = \cap_{a\in S}\, \mathfrak{p}[\frac{X-a}{x}]$. For any polynomial $f \in I$, $f(xX + a)$ belongs to $\mathfrak{p}[\frac{X-a}{x}]$ and thus in particular to $D[\frac{X-a}{x}]$, for any $a \in S$. We then obtain $I = \cap_{a\in S}(\mathfrak{p}[\frac{X-a}{x}] \cap \mathbb{B}_x(D))$. So, if D/xD has finite cardinal, the prime ideal $\mathfrak{P}$ above $\mathfrak{p}$ contains an ideal $\mathfrak{p}[\frac{X-a}{x}] \cap \mathbb{B}_x(D)$, if and only if it contains I.

Proposition 4.3. *Let D be a one-dimensional Noetherian local domain with finite residue field. Let $\mathfrak{m}$ be the maximal ideal of D containing x. Each prime ideal of $\mathbb{B}_x(D)$ above $\mathfrak{m}$, contains an ideal $\mathfrak{m}[\frac{X-a}{x}] \cap \mathbb{B}_x(D)$.*

Proof. First note that D/xD is finite. To see this, as D is a Noetherian local domain, $\mathfrak{m}$ is the radical of xD and there exists an integer α such that $\mathfrak{m}^{\alpha} \subset xD$. The residue field $D/\mathfrak{m}$ is finite, and so are $D/\mathfrak{m}^{\alpha}$ and D/xD.

Denote by S, a set of representatives of D modulo xD. It suffices to show that the ideal $I = \cap_{a\in S}\, \mathfrak{m}[\frac{X-a}{x}]$ is contained in the radical of $\mathfrak{m}\mathbb{B}_x(D)$.

Let $t \in \mathfrak{m}$. As previously, there exists an integer n such that $\mathfrak{m}^n \subset xD$. Let $f \in I$. For all $a \in S$, the polynomial $f(xX + a)$ has its coefficients in $\mathfrak{m}$ and therefore, $f^n(xX + a) \in \mathfrak{m}^n[X] \subset tD[X]$. Dividing by t, we obtain that the polynomial $\frac{1}{t}f^n$ belongs to $\mathbb{B}_x(D)$. Since $t \in \mathfrak{m}$, $f^n \in \mathfrak{m}\mathbb{B}_x(D)$ and f belongs to the radical of $\mathfrak{m}\mathbb{B}_x(D)$. $\square$

Consider D a Noetherian domain and $\mathfrak{p}$ a height one prime containing x with finite residue field. Since we do not have an analogous of Lemma 2.16, we cannot determine all the prime ideals above $\mathfrak{p}$. But, Proposition 4.3 gives the following result:

Corollary 4.4. *Let D be a Noetherian domain, x an element of D. Let $\mathfrak{p}$ be a height one prime, containing x with finite residue field. The prime ideals $\mathfrak{P}$ of $\mathbb{B}_x(D)$ above $\mathfrak{p}$ contain an ideal of the form $\mathfrak{p}D_{\mathfrak{p}}[\frac{X-a}{x}] \cap \mathbb{B}_x(D)$, for an $a \in D$.*

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